Lecture 7

Friday, January 31, 2020 5:58 AM

Power series, revisited.

Consider a power series centered at $a=0$, $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$. $\leftarrow z_1^{\alpha_1} \cdots z_n^{\alpha_n}$

Def 1. The *domain of convergence* D is $z_0 \in \mathbb{C}^n$ s.t. $\sum_\alpha a_\alpha z^\alpha$ converges absolutely in an open nbhd of z_0.

Note: D is open by definition.

Let also B be the set of $z_0 \in \mathbb{C}^n$ s.t. $\exists\, C>0$ w/ $|a_\alpha z^\alpha| = |a_\alpha|\,|z|^\alpha \le C$, $\forall \alpha \in \mathbb{Z}_+^n$.

Recall: $\sum_\alpha a_\alpha z^\alpha$ converges *normally* in $\Omega \subseteq \mathbb{C}^n$ if $\forall K \subset\subset \Omega$ cpt, $\sum_\alpha |a_\alpha| \max_K |z|^\alpha < \infty$. (Also, say $\sum_\alpha a_\alpha z^\alpha$ conv. normally in the compact K.).

Lemma 1. If $w \in B$, then $\sum_\alpha a_\alpha z^\alpha$ converges normally in the polydisk $|z_j| < |w_j|$, $j=1,\dots,n$.

Pf. $\exists\, C$ s.t. $|a_\alpha|\,|w|^\alpha \le C$, $\forall \alpha$. Choose $\tilde{r}_j$ s.t. $\tilde{r}_j < |w_j| = r_j$

Then, $|a_\alpha|\,\tilde{r}^\alpha = |a_\alpha| \cdot r^\alpha \cdot \left(\frac{\tilde{r}}{r}\right)^\alpha \le C \left(\frac{\tilde{r}}{r}\right)^\alpha$.

Since $\frac{\tilde{r}_j}{r_j} < 1 \Rightarrow \sum_\alpha \left(\frac{\tilde{r}}{r}\right)^\alpha < \infty \Rightarrow$ normal convergence in polydisk $\mathbb{D}_r^n = \{|z_j| < r_j, j=1,\dots,n\}$, as claimed. $\boxtimes$

in polydisk $\mathbb{D}_r^n = \{ |z_j| < r_j,\ j=1,\dots,n \}$, as claimed. ☒

<u>Remark:</u> It follows that $u(z) = \sum_\alpha a_\alpha z^\alpha$ is holom. in polydisk $\mathbb{D}_r^n$.

<u>Thm1.</u> $D = \mathring{B}$ = interior of B and $\sum_\alpha a_\alpha z^\alpha$ converges normally to holom. fcn $u(z)$ in D.

<u>Pf.</u> Easy consequence of Lemma 1 (Ex). ☒

<u>Thm2.</u> Let $D^* := \{ \xi \in \mathbb{R}^n : \underbrace{(e^{\xi_1}, \dots, e^{\xi_n})}_{e^{\xi} :=} \in D \}$. Then D^* is open <u>convex</u> subset of $\mathbb{R}^n$ such that if $\xi \in D^*$ and $\eta \in \mathbb{R}$ s.t. $\eta_j \leq \xi_j$, $j = 1, \dots, n$, then $\eta \in D$. Moreover, $z \in D \iff |z_j| \leq e^{\xi_j}$ for some $\xi \in D^*$.

<u>Pf.</u> We note that if $z \in D$, then since D open $\exists$ polydisk $\mathbb{D}_z^n$ centered at z s.t. $\overline{\mathbb{D}_z^n} \subseteq D \Rightarrow \exists\, w \in \mathbb{D}_z^n \subseteq D$ s.t. $|z_j| < |w_j|$, $j = 1, \dots, n$.
All statements in Thm2, except <u>convexity</u>, now follows easily.
We shall show D^* is convex. Define $B^* \subseteq \mathbb{R}^n$ corresponding to $B \subseteq \mathbb{C}^n$ as in thm. It follows from Thm1 that $D^* = \mathring{B}^*$.
It suffices to show B^* is convex. Pick $\xi, \eta \in B^* \Rightarrow z = e^{\xi}, w = e^{\eta} \in B$
$\Rightarrow \exists\, C > 0$ s.t. $|a_\alpha||z|^\alpha = |a_\alpha| e^{\sum \alpha_j \xi_j} \leq C$, $|a_\alpha| e^{\sum \alpha_j \eta_j} \leq C$.
Consider $t \in [0,1]$, and $\tau = t\xi + (1-t)\eta$:

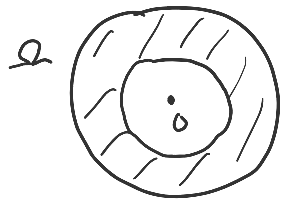
$$|a_\alpha||e^{\tau}|^\alpha = |a_\alpha| e^{\sum_j \alpha_j \tau_j} = |a_\alpha| e^{\sum \alpha_j (t\xi_j + (1-t)\eta_j)} =$$

$$= \left(|a_\alpha| e^{\sum \alpha_j \xi_j}\right)^t \cdot \left(|a_\alpha| e^{\sum \alpha_j \eta_j}\right)^{(1-t)} \leq C^t C^{(1-t)} = C.$$

$\Rightarrow e^{\tau} \in B^{*}$. $\boxtimes$

<u>Def. 2.</u> A <u>Reinhardt domain</u> in $\mathbb{C}^n$ is an open set $\Omega \subseteq \mathbb{C}^n$ s.t. if $z \in \Omega$ then $e^{i\theta} z = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$ $\forall\, \theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$.

<u>Ex.</u> In $\mathbb{C}$, Ω is Reinhardt $\Leftrightarrow$ Ω is $\mathbb{C}$, a disk, punctured disk or annulus.



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